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Dimension of proper G-spaces

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ABSTRACT

In this paper, for G a locally compact group (or a Lie group), we study the relationship between the covering dimensions of a proper G-space X and its orbit space X/G. We prove also that $\dim X = \operatorname{Ind} X$ for every proper G-space X with a metrizable orbit space provided that G is either pro-Lie or σ-compact or has a metrizable quotient group of connected components.

1. Introduction

By a G-space we mean a completely regular Hausdorff space together with a fixed continuous action of a Hausdorff topological group G on it.

The notion of a proper G-space in question was introduced in 1961 by R. Palais [32] with the purpose to extend a substantial portion of the theory of compact Lie group actions to the case of non-compact ones.

Recall that a G-space X is called proper (in the sense of Palais [32, Definition 1.2.2]), if each point of X has a, so-called, *small* neighborhood, i.e., a neighborhood V such that for every point of X, there is a neighborhood U with the property that the set $\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in G. Clearly, if G is compact, then every G-space is proper.

In this paper we investigate relationship between a proper G-space X and its orbit space X/G. We prove the following four main results:

Theorem 1.1. *Let G be a Lie group and X be a proper G-space that admits a G-invariant metric. Then*

$$\dim X/G = \sup\{\dim X_{(H)} - \dim G/H \mid H \text{ is a compact subgroup of } G\}$$

and

$$\dim X = \sup\{\dim X_{(H)}/G + \dim G/H \mid H \text{ is a compact subgroup of } G\}$$

where $X_{(H)} = \{x \in X \mid G_x = gHg^{-1},\ g \in G\}$.

Theorem 1.2. *Let G be a Lie group and X a proper G-space with a paracompact orbit space X/G. Then*

$$\dim X/G \leqslant \dim X.$$

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Here, as usual, dim stands for the covering dimension.

We notice that for G a compact Lie group and X a separable metrizable G-space these results were established in the classical work of R. Palais [31].

Furthermore, it was proved by M.G. Megrelishvili [25] that if G is a compact Lie group then the inequality $\dim X/G \leqslant \dim X$ remains valid for arbitrary Tychonoff G-space X; for a paracompact X the same was proved earlier by S. Deo and S. Tripathi [11].

Theorem 1.3. *Let G be a locally compact group, X a proper G-space with a paracompact orbit space X/G, and $p : X \to X/G$ the orbit map. Then*

$$\dim X \leqslant \dim X/G + \dim p$$

where $\dim p = \sup\{\dim G(x) \mid x \in X\}$.

This inequality for G any compact group and X any Tychonoff G-space was proved earlier by V.V. Filippov [17].

Our fourth result extends a well-known result of B.A. Pasynkov [34] about coincidence of the covering dimension dim with the large inductive dimension Ind:

Theorem 1.4. *Let the locally compact group G be either pro-Lie or σ-compact or have a metrizable quotient group of connected components. If X is a proper G-space with a metrizable orbit space then* $\operatorname{Ind} X = \dim X$.

2. Preliminaries

Throughout the paper all topological spaces and topological groups are assumed to be Tychonoff (i.e., completely regular and Hausdorff). The letter G will denote a locally compact Hausdorff group unless otherwise is stated; by e we shall denote the unity of G.

The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in [10] and [31]. Our basic references on proper group actions are [32,24,2]. For the dimension theory the reader can see, for instance, [15] and [30].

For the convenience of the reader we recall, however, some more special definitions and facts.

If X is a G-space and H a subgroup of G then, for a subset $S \subset X$, $H(S)$ denotes the H-saturation of S, i.e., $H(S) = \{hs \mid h \in H,\ s \in S\}$. In particular, $H(x)$ denotes the H-orbit $\{hx \mid h \in H\}$ of a point $x \in X$. The quotient space of all H-orbits is called the H-orbit space and denoted by X/H.

If $H(S) = S$, then S is said to be an H-invariant set. A G-invariant set will simply be called an invariant set.

For a closed subgroup $H \subset G$, by G/H we will denote the G-space of cosets $\{gh \mid g \in G\}$ under the action induced by left translations.

If X is a G-space and H a closed normal subgroup of G, then the H-orbit space X/H will always be regarded as a G/H-space endowed with the following action of the group G/H: $(gH) * H(x) = H(gx)$, where $gH \in G/H$, $H(x) \in X/H$.

For any $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ of G is called the stabilizer (or stationary subgroup) at x.

The family of all subgroups of G which are conjugate to H is denoted by (H), i.e., $(H) = \{gHg^{-1} \mid g \in G\}$. We will call (H) a G-orbit type (or simply an orbit type); if H is compact then we call (H) a *compact orbit type*. Since $G_{gx} = gGg^{-1}$ for any $x \in X$ and $g \in G$, we have $(G_x) = \{G_{gx} \mid g \in G\}$.

We say that an orbit type (H) occurs in X, if $(G_x) = (H)$ for some $x \in X$. The invariant subset of X consisting of all points $x \in X$ for which $(G_x) = (H)$ is denoted by $X_{(H)}$; the orbit space of $X_{(H)}$ is denoted by $\widetilde{X}_{(H)}$.

A compatible metric ρ on a metrizable G-space X is called invariant or G-invariant, if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$. If ρ is a G-invariant metric on any G-space X, then it is easy to verify that the formula

$$\widetilde{\rho}\big(G(x), G(y)\big) = \inf\big\{\rho\big(x', y'\big) \bigm| x' \in G(x),\ y' \in G(y)\big\}$$

defines a pseudometric $\widetilde{\rho}$, compatible with the quotient topology of X/G. If, in addition, X is a proper G-space then $\widetilde{\rho}$ is, in fact, a metric on X/G [32, Theorem 4.3.4].

A locally compact group G is called *almost connected* if the quotient group G/G_0 of G modulo the connected component G_0 of the identity is compact.

Such a group has a maximal compact subgroup K, i.e., every compact subgroup of G is conjugate to a subgroup of K [1, Theorem A.5]. The corresponding classical theorem on Lie groups can be found in [20, Ch. XV, Theorem 3.1].

Throughout the paper we are specially interested in the following classes of proper G-spaces:

- G-$\mathcal{M}$ – all metrizable proper G-spaces X that are metrizable by a G-invariant metric,
- G-$\mathcal{P}$ – all proper G-spaces that have a paracompact orbit space.

As we mentioned above, a compatible G-invariant metric on a proper G-space X induces a compatible metric on the orbit space X/G, so G-$\mathcal{M} \subset G$-$\mathcal{P}$.

It is well known that, for G a compact group, the class G-$\mathcal{M}$ coincides with the class of *all* metrizable G-spaces (see [31, Proposition 1.1.12]). A fundamental result of R. Palais [32, Theorem 4.3.4], states that if G is a Lie group, then G-$\mathcal{M}$ includes all *separable* metrizable proper G-spaces. The question of whether the separability can be omitted in this Palais' result, still remains open (even for $G = \mathbb{R}$ and $G = \mathbb{Z}$). We refer to [9] for a further discussion of this interesting problem.

Observe that any proper G-space belonging to G-$\mathcal{P}$ is necessarily paracompact [2, Theorem 1.12], while the converse question is a major open problem [2,9]. Here are some partial results toward this problem: a paracompact proper G-space X belongs to G-$\mathcal{P}$ in each of the following cases:

- X is locally Lindelöff and G is almost connected [1, p. 3] and [9, Proposition 2];
- X is metrizable and locally separable and G is separable [9, Corollary];
- X is a topological group and G is a locally compact subgroup of X acting on it by the rule: $g*x = xg^{-1}$ [8, Corollary 1.5].

In what follows we shall need also the definition of a twisted product $G \times_K S$, where K is a closed subgroup of G, and S a K-space. $G \times_K S$ is the orbit space of the K-space $G \times S$ on which K acts by the rule: $k(g,s) = (gk^{-1}, ks)$. Furthermore, there is a natural action of G on $G \times_K S$ given by $g'[g,s] = [g'g,s]$, where $g' \in G$ and $[g,s]$ denotes the K-orbit of the point (g,s) in $G \times S$. We shall identify S, by means of the K-equivariant embedding $s \mapsto [e,s]$, $s \in S$, with the K-invariant subset $\{[e,s] \mid s \in S\}$ of $G \times_K S$. This K-equivariant embedding $S \hookrightarrow G \times_K S$ induces a homeomorphism of the K-orbit space S/K onto the G-orbit space $(G \times_K S)/G$ (see [10, Ch. II, Proposition 3.3]).

The twisted products are of a particular interest in the theory of transformation groups (see [10, Ch. II, §2]). It turns out that every proper G-space locally is a twisted product. For a more precise formulation we need to recall the following well-known notion of a slice (see [32, p. 305]):

Definition 2.1. Let X be a G-space and K a closed subgroup of G. A K-invariant subset $S \subset X$ is called a K-kernel if there is a G-equivariant map $f: G(S) \to G/K$, called the slicing map, such that $S = f^{-1}(eK)$. The saturation $G(S)$ is called a *G-tubular or just a tubular* set, and the subgroup K will be referred as the slicing subgroup.

If, in addition, $G(S)$ is open in X then we will call S a K-slice in X.

If $G(S) = X$ then S is called *a global* K-slice of X.

It turns out that each tubular set $G(S)$ with a compact slicing subgroup K is G-homeomorphic to the twisted product $G \times_K S$; namely the map $\xi: G \times_K S \to G(S)$ defined by $\xi([g,s]) = gs$ is a G-homeomorphism (see [10, Ch. II, Theorem 4.2]). In what follows we will use this fact without a specific reference.

One of the most powerful results in the theory of topological transformation groups is the following theorem of R. Palais [32, Proposition 2.3.1]:

Theorem 2.2 *(Slice Theorem). Let G be a Lie group and X is a proper G-space. Then for any point $x \in X$, there exists a G_x-slice S in X with $x \in S$.*

This theorem has the following important corollary (see [32, §2.3, Corollary 2]):

Corollary 2.3. *Let G be a Lie group and X is a proper G-space and $x \in X$. Then there exists an invariant neighborhood V of x such that for every $y \in V$, the stabilizer G_y is conjugate to a subgroup of G_x.*

In general, when G is not a Lie group, it is no longer true that a G_x-slice exists at each point of X (see [4]). Generalizing the case of Lie group actions, in [6, Theorem 3.6] the following approximate version of Palais' Slice Theorem 2.2 for non-Lie group actions was proved, which plays a key role in our proofs.

Theorem 2.4 *(Approximate Slice Theorem). Let G be a locally compact group, X a proper G-space and $x \in X$. Then for any neighborhood O of x in X, there exist a large subgroup $K \subset G$ with $G_x \subset K$, and a K-slice S such that $x \in S \subset O$.*

Here by a *large* subgroup we mean a *compact* subgroup $H \subset G$ such that the quotient space G/H is a finite-dimensional manifold.

A version of this theorem, without requiring K to be a large subgroup was obtained earlier in Abels [2] (see also [4] for the case of compact non-Lie group actions).

Proposition 2.5. *Let G be a locally compact group and $X \in G$-$\mathcal{P}$. Then there exist large subgroups $H_i \subset G$, $i \in \mathcal{I}$, and a locally finite cover of X consisting of open G-invariant sets Φ_i such that the closure $\overline{\Phi}_i$ is a G-tubular set with the compact large slicing subgroup H_i.*

Proof. It follows from Theorem 2.4 that X can be covered by a family of open tubular sets $G(S_i)$, $i \in \mathcal{I}$, where each S_i is an H_i-slice corresponding to a (compact) large slicing subgroup $H_i \subset G$.

Let $p : X \to X/G$ denote the orbit map. Since X/G is paracompact and $\{p(G(S_i))\}_{i\in\mathcal{I}}$ is an open cover of X/G, there exists an open locally finite cover $\{O_i\}_{i\in\mathcal{I}}$ of X/G such that the closure $\overline{O}_i$ is contained in $p(G(S_i))$ for every $i \in \mathcal{I}$ (see [14, Remark 5.1.7]).

Denote $\Phi_i = p^{-1}(O_i)$, $i \in \mathcal{I}$. Then, clearly, $\{\Phi_i\}_{i\in\mathcal{I}}$ is a locally finite invariant open refinement of $\{G(S_i)\}_{i\in\mathcal{I}}$; moreover $\overline{\Phi}_i \subset G(S_i)$ for every $i \in \mathcal{I}$.

Consequently, each $\overline{\Phi}_i$ being a G-invariant subset of $G(S_i)$, is itself a G-tubular set with the slicing subgroup H_i, as required. □

Finally we recall two results from the dimension theory which are used in the proof of Theorem 3.10. The first one is the following result due to K. Morita [28] and V.V. Filippov [16,19]:

Theorem 2.6. *Let the product $X \times Y$ be a normal space while X is a locally compact paracompact space. Then*

$$\dim X \times Y \leqslant \dim X + \dim Y.$$

The second one is a classical formula which in its final form appeared in [35]:

Theorem 2.7. *Let G be a locally compact group and H and L closed subgroups of G with $L \subset H$. Then*

$$\dim G/L = \dim H/L + \dim G/H.$$

3. Relationship between dim X and dim X/G

3.1. The case of Lie group actions

In the first part of this section we prove the following:

Theorem 3.1. *Let G be a Lie group and X be a proper G-space that admits a G-invariant metric. Then*

$$\dim X/G = \sup\{\dim X_{(H)} - \dim G/H \mid H \text{ is a compact subgroup of } G\}$$

and

$$\dim X = \sup\{\dim \widetilde{X}_{(H)} + \dim G/H \mid H \text{ is a compact subgroup of } G\}.$$

Below we shall give a sequence of lemmas culminating in a proof of this theorem.

Let us begin with the following result established in [5, Proposition 3.6]:

Lemma 3.2. *Let G be a Lie group. Then it has at most countably many compact orbit types.*

Lemma 3.3. *Let G be a Lie group, $X \in G$-$\mathcal{M}$ and H a compact subgroup of G. Then the set $\bigcup\{X_{(K)} \mid (K) \leqslant (H)\}$ is an open invariant subset of X.*

Proof. Immediate from Corollary 2.3. □

Lemma 3.4. *Let G be a Lie group, $X \in G$-$\mathcal{M}$ and H a compact subgroup of G. Then the set $\bigcup\{X_{(K)} \mid (K) < (H)\}$ is an open invariant subset of X.*

Proof. This follows from Lemma 3.3 since

$$\bigcup\big\{X_{(K)} \bigm| (K) < (H)\big\} = \bigcup_{(K')<(H)} \big\{X_{(K)} \bigm| (K) \leqslant \big(K'\big)\big\}. \qquad \square$$

Lemma 3.5. *Let G be a Lie group, $X \in G$-$\mathcal{M}$. Then for any compact subgroup H of G, one has*:

(1) *$X_{(H)}$ is the intersection of an open invariant subset of X and a closed invariant subset of X,*
(2) *$\widetilde{X}_{(H)}$ is the intersection of an open subset of X/G and a closed subset of X/G,*
(3) *$\widetilde{X}_{(H)}$ is an F_σ set in X/G,*
(4) *$X_{(H)}$ is the countable union of closed invariant subsets of X.*

Proof. Observe that $X_{(H)}$ is the intersection of $\bigcup\{X_{(K)} \mid (K) \leqslant (H)\}$ and the complement of $\bigcup\{X_{(K)} \mid (K) < (H)\}$. To prove (1) it remains to observe that the first set is open invariant due to Lemma 3.3, while the second one is closed invariant due to Lemma 3.4.

(2) is immediate from (1).

(3) is immediate from (2) if we previously observe that X/G is metrizable, and every open subset of a metrizable space is F_σ.

(4) is immediate from (3). □

Proposition 3.6. *Let G be a Lie group and $X \in G\text{-}\mathcal{M}$. Then*

$$\dim X = \sup\{\dim X_{(H)} \mid H \textit{ is a compact subgroup of } G\}$$

and

$$\dim X/G = \sup\{\dim \widetilde{X}_{(H)} \mid H \textit{ is a compact subgroup of } G\}.$$

Proof. Since X is a proper G-space, only compact orbit types may occur in it [32, Proposition 1.1.4]. Then X is the union of all its subsets $X_{(H)}$, where H is a compact subgroup of G. Due to Lemma 3.2, at most countably many compact orbit types may occur in X. Hence X is the countable union of its subsets of the form $X_{(H)}$, yielding that X/G is the countable union of its subsets of the form $\widetilde{X}_{(H)}$. But, due to Lemma 3.5, $X_{(H)}$ and $\widetilde{X}_{(H)}$ are F_σ. Now the desired equalities follow from the countable sum theorem for dim [15, Theorem 3.1.8]. □

The following lemma for G a compact Lie group was proved in [3]:

Lemma 3.7. *Let G be a Lie group and $X \in G\text{-}\mathcal{P}$ has only one orbit type, say, (H). Then*

$$\dim X = \dim X/G + \dim G/H.$$

Proof. By Proposition 2.5, X is covered by a locally finite family of closed tubular sets Φ_i, $i \in \mathcal{I}$. This yields that the orbit space X/G is covered by the locally finite family of closed sets Φ_i/G, $i \in \mathcal{I}$.

Hence, by the locally finite sum theorem (see [15, Theorem 3.1.10]), one has

$$\dim X = \sup_{i\in\mathcal{I}} \dim \Phi_i \quad \text{and} \quad \dim X/G = \sup_{i\in\mathcal{I}} \dim \Phi_i/G.$$

Consequently, it suffices to show that

$$\dim \Phi = \dim \Phi/G + \dim G/H$$

for every member Φ of the cover $\{\Phi_i\}_{i\in\mathcal{I}}$.

Let $\varphi : \Phi \to G/H$ be the slicing map corresponding to the tubular set $\Phi \in \{\Phi_i\}_{i\in\mathcal{I}}$, where H is a compact subgroup of G. In this case Φ is G-homeomorphic to the twisted product $G \times_H S$, where $S = \varphi^{-1}(eH)$ (see Section 2).

Note that if $s \in S$ then $G_s \subset H$ and since G_s is conjugate to H it follows that $G_s = H$ (see [10, Ch. 0, Proposition 1.9]). Thus, H acts trivially on S, so S/H is homeomorphic to S. Consequently, the twisted product $\Phi = G \times_H S$ is just G-homeomorphic to the Cartesian product $G/H \times S$. In particular, the orbit space Φ/G is homeomorphic to S.

Now, since G/H is a manifold, and hence a polyhedron, by [27, Theorem B.V] we have

$$\dim \Phi = \dim G/H \times S = \dim G/H + \dim S,$$

and since $S \cong \Phi/G$, then we get

$$\dim \Phi = \dim G/H + \dim \Phi/G,$$

as required. □

Proof of Theorem 3.1. A simple combination of Proposition 3.6 and Lemma 3.7. □

It follows from Proposition 3.6 and Lemma 3.7 that $\dim X/G \leqslant \dim X$ for every $X \in G\text{-}\mathcal{M}$ whenever G is a Lie group. However, this inequality remains valid also for every $X \in G\text{-}\mathcal{P}$ as the following theorem shows:

Theorem 3.8. *Let G be a Lie group and $X \in G\text{-}\mathcal{P}$. Then* $\dim X/G \leqslant \dim X$.

Proof. Observe that the inequality is valid if G is a compact Lie group (see [11] or [25]). We will reduce the general case in question to the compact one.

By Proposition 2.5, X is covered by a locally finite family $\{\Phi_i\}_{i\in\mathcal{I}}$ of closed tubular sets. Then $\{\Phi_i/G\}_{i\in\mathcal{I}}$ is a closed locally finite cover of X/G.

Now, by virtue of the locally finite sum theorem (see [15, Theorem 3.1.10]), it suffices to show that $\dim \Phi/G \leqslant \dim X$ for every member Φ of the cover $\{\Phi_i\}_{i\in\mathcal{I}}$. Let $\varphi : \Phi \to G/H$ be the slicing map corresponding to the tubular set $\Phi \in \{\Phi_i\}_{i\in\mathcal{I}}$, where H is a compact subgroup of G. In this case Φ is G-homeomorphic to the twisted product $G \times_H S$ with $S = \varphi^{-1}(eH)$, and the orbit space Φ/G is homeomorphic to the orbit space S/H (see Section 2).

Next, since H is a compact Lie group, we can use the inequality $\dim S/H \leqslant \dim S$. Since S is a closed subset of the normal space X, due to monotonicity of the dimension by closed subsets (see [15, Theorem 3.1.4]), we have $\dim S \leqslant \dim X$. Thus, $\dim \Phi/G = \dim S/H \leqslant \dim S \leqslant \dim X$, as required. □

Recall that a paracompact space X is said to be finitistic (see [36, p. 394]), if each open cover of X has an open refinement of a finite order. In [11, Theorem 3.6] the following interesting characterization of finitistic spaces is given: a paracompact Hausdorff space X is not finitistic iff there exists a closed set E in X which can be represented as the disjoint union of subsets $\{E_n\}_{n\in\mathbb{N}}$ such that $\dim E_n > n$ and E_n is both closed and open in E for all $n \in \mathbb{N}$.

Proposition 3.9. *Let G be a Lie group and X a finitistic proper G-space such that X/G is paracompact. Then X/G is a finitistic space.*

Proof. Suppose that X/G is not finitistic. Then by the above characterization, there exists a closed set E of X/G which can be represented as the disjoint union of subsets $\{E_n\}_{n\in\mathbb{N}}$ such that $\dim E_n > n$ and E_n is both closed and open in E. Now if $p : X \to X/G$ is the orbit map then by Theorem 3.8, $\dim E_n \leqslant \dim p^{-1}(E_n)$, for each n. Thus $p^{-1}(E)$ is a closed subset of X which is represented as the disjoint union of its subsets $\{p^{-1}(E_n)\}_{n\in\mathbb{N}}$ such that each $p^{-1}(E_n)$ is both closed and open in $p^{-1}(E)$. Hence, again by applying the above characterization, we conclude that X is not finitistic, a contradiction. □

3.2. The case of non-Lie group actions

If the acting group G is not Lie, then the inequality $\dim X/G \leqslant \dim X$ (and hence, the equality in Theorem 3.1) is not true in general. A classical example of A. Kolmogoroff [23] shows that a compact metrizable 0-dimensional group G can act on a 1-dimensional compact metrizable space such that the orbit space is 2-dimensional. The reader can find a further information about compact group actions which raise dimension in [12,13].

In [17] V.V. Filippov established the inequality $\dim X \leqslant \dim X/G + \dim p$ for any compact acting group G and arbitrary G-space X, where $p : X \to X/G$ is the orbit map and $\dim p = \sup\{\dim G(x) \mid x \in X\}$.

Below we generalize this result to the case of proper actions of arbitrary locally compact groups.

Theorem 3.10. *Let G be a locally compact group and $X \in G$-$\mathcal{P}$. Then*

$$\dim X \leqslant \dim X/G + \dim p.$$

Proof. *Case* 1. Assume that G is almost connected. In this case G has a maximal compact subgroup, say, K (see Section 2).

By Proposition 2.5, X is covered by a locally finite family $\{\Phi_i\}_{i\in\mathcal{I}}$ of closed tubular sets. Then, by virtue of the locally finite sum theorem [15, Theorem 3.1.10], it suffices to show that $\dim \Phi \leqslant \dim X/G + \dim p$ for every member Φ of the cover $\{\Phi_i\}_{i\in\mathcal{I}}$.

Let $\psi : \Phi \to G/H$ be the slicing map corresponding to the tubular set $\Phi \in \{\Phi_i\}_{i\in\mathcal{I}}$, where H is a (compact) large subgroup of G. Since, by the maximality of K, there exists an evident G-map $\xi : G/H \to G/K$, the composition $\varphi = \psi\xi$ is a G-map $\varphi : \Phi \to G/K$. In this case Φ is G-homeomorphic to the twisted product $G \times_K S$, where $S = \varphi^{-1}(eK)$ (see Section 2). Furthermore, by a result of H. Abels [1, Theorem 2.1], Φ is homeomorphic (in fact, K-equivariantly homeomorphic) to the Cartesian product $G/K \times S$.

Then, being closed in X, the set $\Phi \cong G/K \times S$ is a normal space. Since G is locally compact and paracompact (see [29, Lemma 1.1]), we infer that so is the quotient space G/K. Then, according to Theorem 2.6,

$$\dim \Phi = \dim(G/K \times S) \leqslant \dim G/K + \dim S. \tag{1}$$

Since K is compact, according to a result of V.V. Filippov [17],

$$\dim S \leqslant \dim S/K + \dim q \tag{2}$$

where $q : S \to S/K$ is the K-orbit projection and $\dim q = \sup\{\dim K(s) \mid s \in S\}$.

Further, since K is compact, each K-orbit $K(s)$ is homeomorphic to the quotient space K/K_s, where K_s is the K-stabilizer of the point $s \in S$.

Consequently, combining (1) and (2), one obtains

$$\dim \Phi = \dim(G/K \times S) \leqslant \dim G/K + \dim S/K + \sup\{\dim K/K_s \mid s \in S\}. \tag{3}$$

Next, we observe that $\Phi/G \cong (G \times_K S)/G \cong S/K$ (see Section 2), and by Theorem 2.7,

$$\dim G/K + \dim K/K_s = \dim G/K_s.$$

Since $s \in S$ and S is a K-slice, we infer that $K_s = G_s$. Then $G/K_s = G/G_s$ which is homeomorphic to the G-orbit $G(s)$ (see [32, Proposition 1.1.5]). Consequently,

$$\begin{aligned}\dim G/K + \sup\{\dim K/K_s \mid s \in S\} &= \sup\{\dim G/K + \dim K/K_s \mid s \in S\} \\ &= \sup\{\dim G/G_s \mid s \in S\} = \sup\bigl\{\dim G(s) \bigm| s \in S\bigr\} \leqslant \dim p.\end{aligned}$$

This, together with (3), implies that

$$\dim \Phi \leqslant \dim \Phi/G + \dim p.$$

Further, since X/G is a normal space and Φ/G is a closed subset of X/G, due to monotonicity of the dimension by closed subsets (see [15, Theorem 3.1.4]), one has $\dim \Phi/G \leqslant \dim X/G$. This, together with the previous inequality, yields

$$\dim \Phi \leqslant \dim X/G + \dim p,$$

as required.

Case 2. Let G be totally disconnected. Then G is zero-dimensional (see [15, Theorem 1.4.5]). It then follows from Theorem 2.7 that $\dim G/H = 0$ for all closed subgroups H of G. Then

$$\dim p = \sup\bigl\{\dim G(x) \bigm| x \in X\bigr\} = \sup\{\dim G/G_x \mid x \in X\} = 0.$$

Hence, one has to prove that

$$\dim X \leqslant \dim X/G.$$

By Proposition 2.5, X is covered by a locally finite family $\{\Phi_i\}_{i\in\mathcal{I}}$ of closed tubular sets. Consequently, by virtue of the locally finite sum theorem (see [15, Theorem 3.1.10]), it suffices to show that $\dim \Phi \leqslant \dim X/G$ for every member Φ of the cover $\{\Phi_i\}_{i\in\mathcal{I}}$.

Let $\varphi : \Phi \to G/K$ be the slicing map corresponding to a tubular set $\Phi \in \{\Phi_i\}_{i\in\mathcal{I}}$, where K is a large subgroup of G. Then the quotient space G/K is locally connected (in fact it is a manifold). On the other hand, G/K is totally disconnected, and hence, it should be a discrete space. This yields that, if we put $S = f^{-1}(eK)$, then each gS is closed and open in Φ, and Φ is the disjoint union of the sets gS one g out of every coset in G/K. In other words, Φ is just homeomorphic to the product $G/K \times S$, like in the previous case. Consequently, preceding like in Case 1, we get the desired inequality $\dim \Phi \leqslant \dim X/G$.

Case 3. Let G be arbitrary locally compact. By Case 1, we have

$$\dim X \leqslant \dim X/G_0 + \dim p_0 \tag{4}$$

where G_0 is the identity component of G and $p_0 : X \to X/G_0$ is the G_0-orbit map.

Since G/G_0 is a totally disconnected locally compact group and the induced action of G/G_0 on the G_0-orbit space X/G_0 is proper (see [32, Proposition 1.3.2]), according to Case 2, we have

$$\dim X/G_0 \leqslant \dim \frac{X/G_0}{G/G_0}. \tag{5}$$

Next we observe that for any $x \in X$, the G_0-orbit $G_0(x)$ is a closed subset of the G-orbit $G(x)$ (see [32, Proposition 1.1.4]) which is a normal space. Hence, due to monotonicity of the dimension by closed subsets (see [15, Theorem 3.1.4]), one has $\dim G_0(x) \leqslant \dim G(x)$. This yields that $\dim p_0 \leqslant \dim p$.

Now, since

$$\frac{X/G_0}{G/G_0} \cong X/G$$

it then follows from (4) and (5) that

$$\dim X \leqslant \dim X/G + \dim p.$$

This completes the proof. □

The following result for G an almost connected group is proved in [8]:

Corollary 3.11. *Let X be a paracompact topological group and G a locally compact subgroup of X. Then*

$$\dim X \leqslant \dim X/G + \dim G.$$

Proof. Immediate from Theorem 3.10 since the natural action of G on X given by the formula $g * x = x \cdot g^{-1}$ is proper and the quotient space X/G is paracompact (see [8]). □

4. Coincidence of the dimensions dim and Ind

In this section we will prove the following theorem which for compact group actions was established by B.A. Pasynkov [33]:

Theorem 4.1. *Let the locally compact group G be either pro-Lie or σ-compact or have a metrizable quotient group of connected components. If X is a proper G-space with a metrizable orbit space then* $\operatorname{Ind} X = \dim X$.

Here, as usual, σ-compactness of G means that G has a countable covering consisting of compact subsets. A locally compact group is called pro-Lie [22, p. 2, Definition 1] if it has arbitrary small closed normal subgroups H such that G/H is a Lie group. It is worth to mention that each almost connected group, as well as each locally compact abelian group, is pro-Lie (see e.g., [22, Ch. 1, p. 2 and Ch. 3, Example 3.4]).

The proof of Theorem 4.1 is based on the following result proved in [7]:

Theorem 4.2. *Let G be a locally compact group and X a proper G-space such that all the orbits in X, as well as the orbit space X/G, are metrizable. Then X is metrizable. Moreover, there exists a compatible G-invariant metric on X.*

Note that this theorem for G a compact metrizable group was first obtained by B.A. Pasynkov [33, Theorem 2], and for G any compact group by V.V. Filippov [18, Theorem (b)].

Corollary 4.3. *Let G be a locally compact group and H a closed normal subgroup of G such that G/H is metrizable. Then for each proper G-space X with a metrizable orbit space X/G, the H-orbit space X/H is metrizable too. Moreover, there exists a compatible G/H-invariant metric on X/H.*

Proof. The H-orbit space X/H is a proper G/H-space with regard of the induced G/H-action (see [32, Proposition 1.3.2]). We aim at applying Theorem 4.2 to X/H, so let us check that its hypotheses are satisfied. Indeed, since the acting group G/H is metrizable, all the G/H-orbits in X/H are so (this is because all the orbits in X/H are homeomorphic to the quotients of G/H [32, Proposition 1.1.5]). Besides the G/H-orbit space of X/H is just homeomorphic to X/G, and hence, it is metrizable by the hypothesis. Therefore, by Theorem 4.2, X/H is metrizable by a G/H-invariant metric. □

Not every locally compact group satisfies the hypothesis of this corollary. Namely, it is observed in [37, p. 412] that there exist locally compact totally disconnected non-metrizable groups without non-trivial closed normal subgroups.

However, below we show two fairly vast classes of locally compact groups which satisfy the hypothesis of Corollary 4.3:

Proposition 4.4. *A locally compact group G has a compact normal subgroup $H \subset G$ such that the quotient group G/H is metrizable in each of the following cases*:

(1) *G is a pro-Lie group,*
(2) *G is a σ-compact group.*

Proof. (1) Since G is pro-Lie, it has arbitrary small closed normal subgroups H such that the quotient group G/H is Lie, and hence, metrizable. By local compactness of G the subgroup H can be chosen to be compact.

(2) See [21, Theorem 8.7]. □

By a simple combination of Corollary 4.3 and Proposition 4.4 we obtain the following:

Corollary 4.5. *Let the locally compact group G be either pro-Lie or σ-compact, and let X be a proper G-space such that the orbit space X/G is metrizable. Then there exists a compact normal subgroup $H \subset G$ such that G/H is a metrizable group and the H-orbit space X/H is metrizable by a G/H-invariant metric.*

Corollary 4.6. *Let G be a locally compact group such that the quotient group G/G_0 of G modulo the connected component G_0 of the identity is metrizable. If X is a proper G-space such that the orbit space X/G is metrizable, then there exists a compact subgroup $H \subset G$ such that the H-orbit space X/H is metrizable.*

Proof. We aim at applying Corollary 4.3 twice. First we apply it to G and G_0 and get that the G_0-orbit space X/G_0 is metrizable.

Next, observe that X is a proper G_0-space with respect to the induced G_0-action. According to a result of Yamabe [38], every identity neighborhood of G_0 contains a compact normal subgroup $H \subset G_0$ such that the quotient G_0/H is a Lie group; in particular G_0/H is metrizable. Since the G_0-orbit space X/G_0 is metrizable, one can apply Corollary 4.3 once more and get the metrizability of the H-orbit space X/H, as required. □

Remark 4.7. A space X on which a compact group K acts in such a way that the orbit space X/K is metrizable is called, by B.A. Pasynkov [34], *almost metrizable*.

In this terminology, Corollaries 4.5 and 4.6 imply that the proper G-space X in question is almost metrizable.

Proof of Theorem 4.1. It follows from Corollaries 4.5 and 4.6 that there exists a compact subgroup H of G such that the H-orbit space X/H is metrizable. Thus, X is almost metrizable and then the required equality follows from Pasynkov's result [34, Theorem]. $\square$

In Corollary 4.5, if X is a topological group and G a locally compact subgroup of X, then additional restrictions on G are unnecessary. First we recall the following lemma:

Lemma 4.8. *([7]) Let G be any topological group, H a subgroup of G and N a compact subgroup of H. Suppose that G/H and H/N both are metrizable. Then G/N is metrizable too. Moreover, there exists a G-invariant metric on G/N.*

Corollary 4.9. *Let X be a topological group and G a locally compact subgroup of X. If the quotient space X/G is metrizable, then there exists a compact subgroup $H \subset G$ such that the quotient space X/H is metrizable.*

Proof. It is well known that every locally compact group admits an open almost connected subgroup, say U (see [26, Ch. II, §2.3]). By the above quoted theorem of Yamabe [38], U has a compact normal subgroup such that U/H is a Lie group, and hence, is metrizable. Since the quotient G/U is also metrizable (it is a discrete space), it then follows from Lemma 4.8 that G/H is metrizable.

Next, since X/G is metrizable, applying once again Lemma 4.8 we get the metrizability of X/H, as required. $\square$

Corollary 4.10. *Let X be a topological group and G a locally compact subgroup of X such that the quotient space X/G is metrizable. Then* $\dim X = \operatorname{Ind} X$.

Proof. Immediate from Corollary 4.9 and Pasynkov's result [34, Theorem]. $\square$

In connection with the results of this section, the following questions seem worthy of study:

Question 1. Let G be a locally compact group and X a proper G-space such that the orbit space X/G is metrizable. Assume that H is a compact subgroup of G such that G/H is metrizable. Is then the H-orbit space X/H metrizable?

Question 2. Let G be a locally compact group and X a proper G-space such that the orbit space X/G is metrizable. Does there exist a compact subgroup H of G such that the H-orbit space X/H is metrizable?

Obviously, Question 2 is a weaker version of Question 1. As one can see from the proof of Theorem 4.1, a positive answer to Question 2 yields validity of Theorem 4.1 for any locally compact group G.

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